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A MATHEMATICAL STUDY OF DECISIONS IN A DICTATORSHIP

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A MATHEMATICAL STUDY
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DECISIONS IN A DICTATORSHIP

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33

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13. ABSTRACT

The optimal location of a benevolent dictator in a multidimensional issue space is examined under various ℓ_p norms for two objective functions -- minimizing total utility loss to the citizens and minimizing the maximim utility loss.

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INTRODUCTION

Since the work of Hotelling [14] a growing number of political theorists have been explicitly employing "spatial" techniques to investigate various aspects of electoral behavior. Perhaps the work in this area which is best known to political scientists is that of Downs [8]. More recently there has been considerable theoretical work done expanding the Downs approach. Much of this research has been made possible by exploiting the increased deductive power of mathematics. One of the greatest virtues of expressing political theories in a mathematical language is the relative ease with which previous results may be generalized. The advantage of generality in empirical theory is, of course, that the more general the statement the less it assumes to be true of reality. As an example, the analysis done by Downs assumes (among other things) that every voter's "most preferred position" can be located somewhere in a one dimensional issue space. As Stokes [28] has pointed out, there is strong empirical evidence that no such single dimension exists. A way around this objection is then to assume that each citizen's "most preferred position" can be located somewhere in an "n" dimensional space. Theory statements derived under this assumption are no longer dependent upon the (empirical) existence of a one dimensional issue space. In summary, therefore, the more general the assumptions, the less must be true in any reality for the theory to be true of that reality.

The major concern of this paper will be to generalize a number of the assumptions and results of Davis, Hinich and Ordeshook [5] . The nature of these assumptions (as well as our generalizations) will be explicated in succeeding sections of this paper. The specific problem which will be

focused upon will be that of where in some n-dimensional issue space a benevolent dictator ought to locate if he (she) wishes to maximize some well defined objectives (such as minimize citizens total utility loss or minimize maximum citizen utility loss). In other papers [34] [29] [35], this analysis is extended to candidate positions under majority rule.

The benevolent dictator problem is especially tractable since it is formally very similar to problems arising in Operations Research under the heading of "plant location problems." This problem deals with the question of where in some space (generally two-dimensional) a plant ought be located if it is to minimize the cost of shipping inputs from various points in that space to the plant. Kuhn [19] sets up this problem as:

$$\min_{X} \sum_{i=1}^{m} w^i \, || X - d_i ||$$

where $X \equiv$ vector variable in the space denoting the location of the plant

$w^i \equiv$ cost per unit distance of shipping a (known) quantity of input i from d_i

$|| X - d_i || \equiv$ the (generally Euclidean) distance from X to d_i.

As will be shown, this is very much analagous to the dictator problem where X would be the dictator's location, w^i the i_{th} citizen's loss function and $|| \cdot ||$ the measure of distance of the i_{th} person from the dictator's position.

This formal similarity is very nice as it allows us to directly reinterpret much of "plant location theory" directly into political science terms*. An important side benefit of this is that this reinterpretation

*In model-theoretic terms, a model for plant location theory can be found in political science.

motivates new theorems which can be added to the corpus of both "plant location theory" and "political theory." The wedding of "plant location theory" with "spatial analysis" is interesting both in terms of the advances it permits in political theory and in terms of new mathematical results.

2

1. Spatial Theory and the Dictator Problem

In recent years various articles (see, for example [5], [24]) have formulated and analyzed political processes using a spatial theory approach. We will not give the background or assumptions of this approach here (see [5] for a good introduction), but instead we will simply review the notation and formulations that have been established elsewhere [5], [24].

Consider a situation in which there are n issues. Then following [5] we define*:

$x_i \equiv (x_i^1, x_i^2, \ldots, x_i^n)' \equiv$ the vector that characterizes the position of citizen i for each of the m citizens ($i = 1, \ldots, m$) on all n issues, $x_i \varepsilon R^n$, and R^n will be called the issue space (e.g., R^n is the usual space of real n-tuples);

$\theta \equiv (\theta^1, \theta^2, \ldots, \theta^n)' \equiv$ the vector of positions advocated by the dictator**, $\theta \in R^n$;

$L_i(\theta) \equiv$ the loss citizen i sustains from the dictator's position θ;

$A \equiv$ a positive definite symmetric nxn matrix such that $(x_i - \theta)' A (x_i - \theta)$ generates indifference

---

*The notation "'" denotes the transpose of a vector. Unlike [5] we use superscripts to denote components of a vector. The distinction between superscripts and exponents will be clear from the context.

**In a democracy analysis this is commonly used to denote the position of a candidate.

contours* for the loss function of citizen i about his location x_i;

$\phi(\cdot) \equiv$ a monotonically increasing function of its argument such that $L_i(\theta) = \phi((x_i - \theta)' A (x_i - \theta))$.

With respect to this notation Davis, Hinich, and Ordeshook (see page 441 of [5]) consider the problem of "an omniscient and beneficent dictator faced with the task of selecting the 'best' policies** for his country." In particular, the dictator may simply wish to "minimize the total utility loss of the society." In this case with respect to the above notation his problem becomes

$$(1.1)\quad \underset{\theta}{\text{Min}} \sum_{i=1}^{m} \phi((x_i - \theta)' A (x_i - \theta)).$$

Note that the above formulation implicitly assumes that such a dictator weights each citizen identically. Following [5], we now generalize this to the possibility of unequal weights for the citizens. First we define:

$f(x) \equiv$ "the electorate's preference density" that gives the distribution of the citizens x_i in the issue space;

$w(x) \equiv$ a "weighting function" that assigns a positive weight $w(x_i)$ to each citizen i.

Now we can state the more general dictator problem:

$$(1.2)\quad \underset{\theta}{\text{Min}} \sum_{i=1}^{m} w(x_i)\, \phi((x_i - \theta)' A (x_i - \theta)).$$

Note that if all the weights $w(x_i)$ are equal, then problem (1.2) reduces to problem (1.1).

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*See [34] or any good economics text for a definition of indifference contours.

**As we shall see, "best" is a very ambiguous criterion by which to analyze policies.

3

For simplicity, we will choose the mean of "the electorate's preference density" $f(x)$ as the origin of our coordinate system. Then, following [5], we make the following assumptions:

(1.3) (i) $f(x)$ is symmetric about the origin (e.g. about its mean);

(ii) $w(x)$ is symmetric about the origin (e.g. about the mean of $f(x)$).

Assumption (i) simply implies that for every citizen i at a position x_i there exists a citizen k at a position $-x_i$. Assumption (ii) simply means that $w(x) = w(-x)$.

It is interesting to note that Davis, Hinich, and Ordeshook [5] distinguish two general forms of the weighting function $w(x)$. "First, the beneficient dictator might assign more importance to those in the 'middle' than to those who held extreme positions, and in this instance we say that $w(x)$ is unimodal. Second, the dictator might weight 'liberals' and 'conservatives' more heavily than 'moderates' and in this instance $w(x)$ is termed not unimodal." If one carries this weighting of "extremists" to the extreme (e.g. to the point where the dictator is only concerned about them) then we essentially get the class of problems which we consider in section 4. Otherwise, the problem will be a special case of the type that we consider in section 3.

With respect to problem (1.2), we can choose the eigenvectors of A as a basis for R^n with an associated scale change on each of the axes according to the square root of the respective eigenvalue* to get the following

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*See pages 432 to 434 of [5] and especially footnote 18 on page 434 for more discussion on this point.

problem which is equivalent to (1.2):

$$(1.4) \quad \min_{\theta} \sum_{i=1}^{m} w(x_i)\, \phi\,((x_i - \theta)'\,(x_i - \theta)) \quad .$$

Differentiating (1.4) with respect to θ^k we get the first order necessary conditions*:

$$\sum_{i=1}^{m} w(x_i)\, [\dot{\phi}((x_i - \theta)'\,(x_i - \theta))]\, [(\theta^k - x_i^k)] = 0$$

for $k = 1, \ldots, n$. For $\theta = 0$ this reduces to

$$(1.5) \quad \sum_{i=1}^{m} w(x_i)\, [\dot{\phi}\ (x_i'\, x_i)]\, [x_i^k] = 0 \text{ for } k = 1, \ldots, n.$$

By the assumptions in (1.3) the conditions (1.5) are certainly true. Thus, the mean of $f(x)$ is certainly a possible candidate as the optimal location for the dictator (e.g. it is a stationary point). To show that it is indeed the optimal location further analysis is required.

In the case where the loss functions are marginally increasing one can argue via convexity that such a point will always be the optimal dictator location (e.g. the global solution). Otherwise, this cannot be guaranteed unless we have even more special situations. In particular, Davis, Hinich, and Ordeshook [5] claim that even when loss functions may be marginally increasing and marginally decreasing the optimal location is still at the mean if we further assume that both $f(x)$ and $w(x)$ are unimodal**. Although they don't justify this claim, it probably results from some argument involving strict quasiconvexity.

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*$\dot{\phi}$( ) means the derivative of ϕ.

**In the discrete case where the number of citizens is finite, the definition of unimodality of $f(x)$ given on page 438 of [5] is not clear. In particular, if each citizen's location is distinct, then each x_i may be viewed as a mode of $f(x)$.

DICTATOR'S PREFERENCE

	Loss Functions Marginally Increasing Only		Loss Functions Marginally Increasing and Marginally Decreasing	
f(x)	w(x) Unimodal	Otherwise	w(x) Unimodal	Otherwise
Symmetric Unimodal	Mean	Mean	Mean	No General Solution
Symmetric Bimodal	Mean	Mean	No General Solution	No General Solution

TABLE 1.1

4

Finally, we summarize the results of the Hinich, Davis, Ordeshook analysis of the dictator problem in Table 1.1. Based on this table they conclude:

PLACE TABLE 1.1 ABOUT HERE

"The point here, however, is that in a variety of ethical assumptions arbitrarily assigned, the mean appears to be a desirable point. Accordingly, contrary to the responsible parties doctrine, forces which cause both party platforms to converge toward the mean, rather than recognizing differences in opinion, are not necessarily 'bad' and, in the majority of the above cases, are positively 'good' if one is willing to accept the assumptions."

5

2. Metrics and Norms

A central assumption of the "spatial approach" is that each citizen's "loss" from a particular position of the dictator is dependent upon the (citizen's) perceived dissimilarity between the dictator's spatial location and the location of the citizen. This perceived dissimilarity may be treated as having the properties of a distance metric.

More formally, the problem is to measure the distance between any two points ξ_1 and ξ_2 in R^n. In general, the problem is to define a function $\rho(\xi_1, \xi_2)$ from pairs of points ξ_1, ξ_2 into R^1 such that the resulting scalar measures the distance between the points ξ_1 and ξ_2. If such a distance function satisfies the following properties, then it is called a metric (see [13], or [17] for further discussion).

(2.1) (i) $\rho(\xi_1, \xi_2) \geq 0$ for all $\xi_1, \xi_2 \in R^n$ (e.g. distance is always non-negative);

(ii) $\rho(\xi_1, \xi_2) = 0$ iff $\xi_1 = \xi_2$ (e.g. distance between two points is zero iff the two points are identical);

(iii) $\rho(\xi_1, \xi_2) = \rho(\xi_2, \xi_1)$ (e.g. the distance from one point to a second is equal to the distance from the second to the first);

(iv) $\rho(\xi_1, \xi_2) \leq \rho(\xi_1, \xi_3) + \rho(\xi_3, \xi_2)$ (e.g. the distance from one point to another is always less than or equal to the sum of the distance from these points to a third point).

Property (iii) is often called the symmetry property while property (iv) is the triangle inequality. The metric is a general distance measure which

includes many more specific distance measures as special cases.

The general properties of a metric are satisfied by many distance functions. Perhaps the most familiar of these is the Euclidean distance function. Indeed all the work done by Davis, Hinich, and Ordeshook assumes the Euclidean metric for all citizens. However empirical work done by perceptual psychologists [2], [10], [16], [26] suggests very strongly that other metrics may be appropriate for best ordering wide ranges of data.

If this be the case, several classes of problems become important. First, what of the cases where all citizens have the same metric but that metric is of some other form than the Euclidean. Second, and more generally, what if the distance measure "used" by each citizen satisfies the properties of a metric, but no restrictions are placed as to what specific metric any particular citizen employs. In order to investigate these problems it is first necessary to provide a general discussion of distance measures and define several potentially interesting non-Euclidean measures.

Example 2.1

Define

$$\rho(\xi_1, \xi_2) = \begin{cases} 1 \text{ if } \xi_1 \neq \xi_2 \\ 0 \text{ if } \xi_1 = \xi_2 \end{cases} .$$

Since properties (i), (ii), (iii), and (iv) are true, $\rho(\xi_1, \xi_2)$ is clearly a metric. A citizen x_1 having such a metric in an issue space is one who can distinguish whether the dictator agrees with him (e.g. $\theta = x_1$) or disagrees with him (e.g. $\theta \neq x_1$) but can make no distinctions as to the extent of disagreement.

6

Unlike a metric which maps $R^n \times R^n$ into R^1*, we now define a norm which maps R^n into R^1 (see [17] or [22] for a further discussion). In particular, a norm is a function $||\cdot||$ with the following properties:

(2.2) (i) $||\xi|| \geq$ for all $\xi \in R^n$;

(ii) $||\xi|| = 0$ iff $\xi = 0$;

(iii) $||\xi|| = ||-\xi||$;

(iv) $||\overline{\xi}_1 + \overline{\xi}_2|| \leq ||\overline{\xi}_1|| + ||\overline{\xi}_2||$ for $\overline{\xi}_1, \overline{\xi}_2 \in R^n$**;

(v) $||\alpha\xi|| = |\alpha| \, ||\xi||$ for any scalar α .

With the exception of property (v), the properties of a metric are similar to those of a norm. In particular, given a norm $||\cdot||$ we can define a metric

$$\rho(\xi_1, \xi_2) \equiv ||\xi_1 - \xi_2||$$

and verify the corresponding metric properties by letting $\xi \equiv \xi_1 - \xi_2$, $\overline{\xi}_1 \equiv \xi_1 - \xi_3$, and $\overline{\xi}_2 \equiv \xi_3 - \xi_2$. Thus, we can think of a norm as a special case of a metric. On the other hand, if the metric $\rho(\cdot,\cdot)$ satisfies the additional properties (see page 50 of [17]).

(2.3) (v) $\rho(\xi_1 + \xi_3, \xi_2 + \xi_3) = \rho(\xi_1, \xi_2)$

(vi) $\rho(\alpha\xi_1, \alpha\xi_2) = |\alpha| \rho(\xi_1, \xi_2)$ for any scalar α,

then the metric $\rho(\cdot,\cdot)$ can be used to define a norm

$||\xi|| \equiv \rho(\xi_1, 0)$

---

*$R^n \times R^n$ is the Cartesian product of R^n with R^n

**$\overline{\xi}_1$ and $\overline{\xi}_2$ are arbitrary vectors in R^n and this notation is used to eliminate confusion with the definition of a metric in subsequent discussion.

There are many different types of norms of which we will now give some examples.

Example 2.2: City Block Norm

Consider a norm defined as

$$||\xi||^{(1)} \equiv \sum_{j=1}^{n} |\xi^j| \quad \text{where } \xi \in R^n$$

(e.g. the superscript (1) is used to denote this type of norm).

Such a norm is often called the city block norm, the Manhattan norm, or the ℓ_1 norm. The reason for the former names is that in various urban areas (e.g. Manhattan) the city streets are in the north-south and east-west directions. Given a north-south and east-west coordinate system, a distance vector ξ has a length given by the sum of absolute values of the coordinates. For example, see Figure 2.1. The reason for it being called the ℓ_1 norm will become clear in Example 5.

Besides the above interpretation, the ℓ_1 norm has an interesting interpretation in distance perceptions in an issue space. Consider a citizen's positon at a point x_i and a dictator's position at the point θ. Then

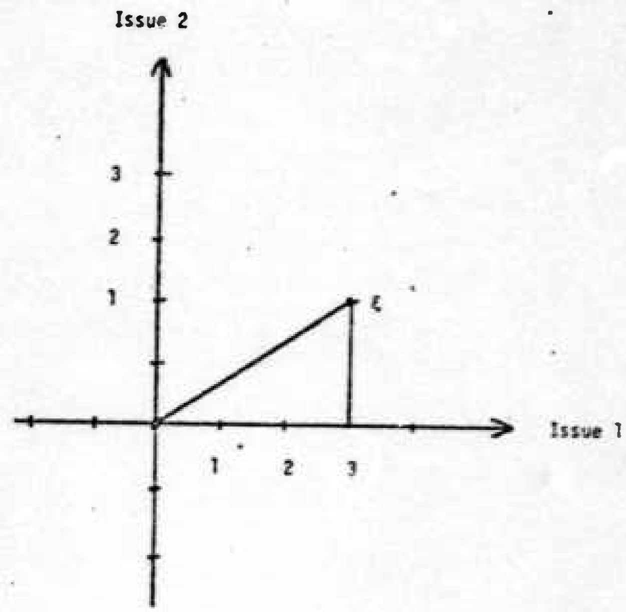
$$||\theta - x_i||^{(1)} = \sum_{j=1}^{n} |\theta^j - x_i^j|$$

and the citizen views the distance from him to the dictator as the sum of the distances that their views vary on each issue.

PLACE FIGURE 2.1 ABOUT HERE

Since the empirical work of Attneave [2], the city block norm has been of some interest to psychologists in analyzing perceptual data. Indeed the argument has been made [2] that the city block norm rather than the

7



$$||\xi||^{(1)} = |3| + |2| = 3 + 2 = 5$$

$$||\xi||^{(2)} = [(3)^2 + (2)^2]^{1/2} = [13]^{1/2}$$

$$||\xi||^{(\infty)} = \text{Max } \{ |3|, |2| \} = 3$$

Figure 2.1

Euclidean should be considered as fundamental for ordering perceptual data, since subjects seemed to judge dissimilarities in geometric stimuli by independently judging differences in components (dimensions). Further, the city block norm has nice additive properties not possessed by the Euclidean norm. Indifference contours of this type of individual are given in Figure 2.2. Note the diamond shape of these contours.

<u>Example 2.3</u>

Consider a norm defined as

$$||\xi||^{(\infty)} \equiv \underset{j=1,\ldots,n}{\mathrm{Max}} \{|\xi^j|\} \quad \text{where } \xi \in R^n$$

(e.g. the superscript (∞) is used to denote this type of norm).

Such a norm is often called the sup norm, the Tchebycheff, or the ℓ_∞ norm. Unlike the previous norm, the ℓ_∞ norm has an interpretion that the distance from dictator to a citizen equals the maximum of the differences in positions over all issues (see Figure 2.1). Mathematically, we write this as

$$||\theta - x_i||^{(\infty)} = \underset{j=1,\ldots,n}{\mathrm{Max}} \{|\theta^j - x_i^j|\} .$$

This norm might well be of special interest in political science. In particular, a citizen who measures distance under such a norm would ignore a dictator's position on all issues but the one which achieved maximum disparity (e.g. Vietnam).

PLACE FIGURE 2.2 ABOUT HERE

As shown in Figure 2.2 the indifference contours for this type of norm are box-like. Since a rotation of the box is a diamond, one might

8

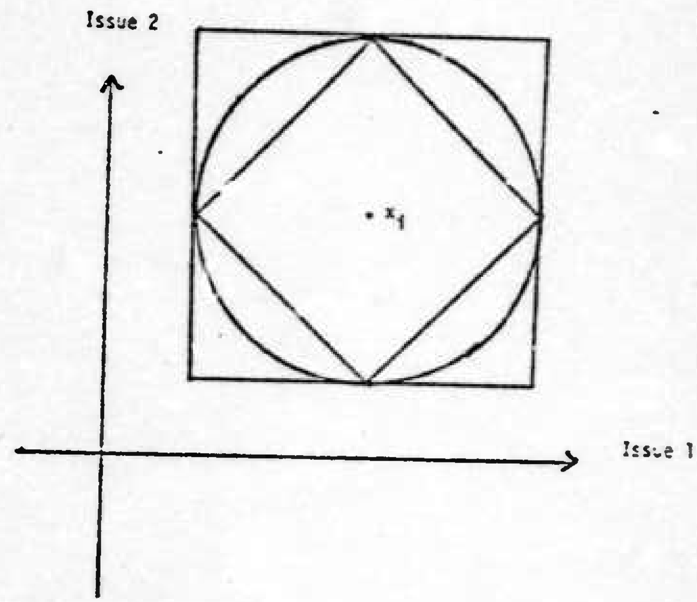


Figure 2.2

suspect that the mathematical structure of ℓ_∞ persons and the ℓ_1 person are related. We will now investigate this relationship between these two norms.

Theorem 2.1 In R^2 the ℓ_1 norm and the ℓ_∞ norm are equivalent under a change of variables (e.g. a 45 degree rotation).

For a proof see [35].

Just as significant as the above theorem is the fact that the norms are not equivalent in R^n for $n \geq 3$. In R^3, for example, the diamond will have 6 extreme points (e.g. corners) while a box has 8 extreme points. Thus, no matter how much one rotates the two figures, they will never become "equivalent."

Example 2.4

As we all know, the Euclidean norm is defined as

$$||\xi||^{(2)} \equiv \left[\sum_{j=1}^{n} (\xi^j)^2 \right]^{1/2}$$

where the superscript (2) is used to denote it. Alternatively, we will refer to the Euclidean norm as the ℓ_2 norm. Except for property (iv), it is easy to verify that the properties of a norm are satisfied (Property (iv) is a special case of the Minkowski inequality--see page 31 of [22]). Also, it is clear that the indifference contours of this norm are circles (see Figure 2.2).

Example 2.5

In a more general sense we define an ℓ_p norm for $p \geq 1$ as

$$||\xi||^{(p)} \equiv \left[\sum_{j=1}^{n} |\xi^j|^p \right]^{1/p}$$

where the superscript (p) is used to denote the particular ℓp norm. As

before, the proof of property (iv) of norms follows from the Minkowski inequality. Note that Examples 2.2, 2.3, and 2.4 are special cases when $p = 1, 2,$ and ∞ respectively. The indifference contours are illustrated in Figure 2.1.

Example 2.6

Consider a norm defined as

$$(2.4) \qquad ||\xi|| \equiv (\xi' A \xi)^{1/2}$$

where A is a positive definite matrix. This is the norm used in much of the previous spatial theory analyses (as in [5], [24]).

See Householder [15] for a proof of the fact that this does indeed satisfy the properties (2.2) of a norm. For our purposes, it is sufficient to point out that since A is positive definite one can choose an eigenvector basis such that A can be represented as a diagonal matrix whose elements along the diagonal are its strictly positive eigenvalues (see [5] for a further discussion). By changing the scale along this new basis according to the square root of these eigenvalues, the norm (2.4) becomes equivalent to the Euclidean norm (Footnote 18 of [5] also points this out). Thus, the norm (2.4) gives no new mathematical generality to the problem than does a formulation under the simple Euclidean norm.

In summary, this section has introduced various abstract distance functions and has discussed how different behaviors can be related to different functions. Metrics characterized a very general class of different functions which includes norms as an important subclass. In particular, ℓ_p norms give a characterization of a whole class of norms which include the Euclidean norm as a special case. The Euclidean norm case is of course the distance measure which has been considered in previous spatial theory analysis. In

9

Figure 2.2 we have illustrated the shapes of indifference contours characterized by ℓ_p norms. Although this represents a broad behavior class, the fact that there is a one-to-one correspondence between norms and symmetric convex indifference contours* (see page 132 of [25] for further discussion) indicates that much broader classes of behavior other than ℓ_p norms can be considered by studying problem structures using norms.

*An indifference contour is convex when the set it contains is convex. Convex sets is one of the items to be discussed in the Appendix.

10

3. Minimizing Total Utility Loss

As indicated in section 1, the spatial analyses of dictatorships have assumed that the dictator wishes to minimize (weighted) total utility loss to the citizens as formulated in problem (1.2).

In this section we will use the concepts of metrics and norms to generalize this problem structure and give results of analysis of these structures. It will also be shown how problem (1.2) is a very special case of our general formulation.

The most general formulation that we consider is:

$$\text{(T1)} \quad \min_{\theta} \sum_{i=1}^{m} C_i\left(\rho_i\left(x_i, \theta\right)\right)$$

where

$C_i(\cdot) \equiv$ a non-decreasing function which gives a loss measure for each of the m citizens ($i=1, \ldots, m$);

$\rho_i(x_i, \theta) \equiv$ a metric that measures the distance that citizen i perceives to exist between his position x_i and the dictator's position θ (e.g. the type of metric is allowed to depend on the citizen).

Unfortunately, not much can be said about the general problem (T1). We, therefore, consider the following special case (i.e. more structural problem).

$$\text{(T2)} \quad \min_{\theta} \sum_{i=1}^{m} w_i\, \rho_i\left(x_i, \theta\right)$$

The weights w_i are positive scalar constants. In particular, (T2) results as a special case of (T1) when one lets $C_i(z) \equiv w_i z$. Again we will not give any results, but instead we give another special case of (T1).

$$(T3)\quad \min \sum_{i=1}^{m} C_i\,(\rho\,(x_i,\theta))$$

Unlike (T1), note that (T3) assumes that all the metrics are of the same type (e.g. the type of metric does not depend on the citizen). Such an assumption may indeed be true for various organizations or groups (e.g. people who "think alike" may quite naturally tend to gravitate together). At any rate, the assumption in [5], [24] that the matrix A is independent of each citizen x_i is a special case of the above assumption.

Finally we give a problem, that is a special case of (T2) and (T3), about which we can state a result.

$$(T4)\quad \min_{\theta} \sum_{i=1}^{m} w_i\,\rho(x_i,\theta)$$

Note that this problem is a special case of (T2) in that (T4) assumes that the same type of metric ρ applies to all citizens. On the other hand, it follows from (T3) by assuming that $C_i(z) = w_i z$ for each i.

Theorem 3.1 If $w_k \geq \sum_{\substack{i=1\\ i\neq k}}^{m} w_i$, then $\theta^* = x_k$.

Proof (see page 30 of [37])

A point x_k that satisfies the property that $w_k \geq \sum_{\substack{i=1\\ i\neq k}}^{m} w_i$ is said to be a singular point. Thus the theorem (often called the majority theorem) states that a singular point is an optimal location for the dictator who desires to minimize total utility loss. This is true regardless of the type of metric being considered (e.g. we only assumed that all metrics were of the same unspecified type). The weights w_i can be viewed as either a linear approximation to the loss function $C_i(\cdot)$ of the corresponding citizen x_i or as the dictator's perception of this linear approximation. In the latter case, the weight w_i can be considered as an index of the relative importance

that the dictator views each point x_i. In this case we can think of w_i as a measure of the power of a group located at the point x_i. Thus, the benevolent dictator, who perceives the power of one group to be greater than or equal to the sum of the power of all other groups, will adopt a position identical to that most powerful group. Note that this is true regardless of the relative location x_i of the groups (e.g. citizens).

We now consider the following special case of problem (T1).

$$(T5)\quad \min_{\theta} \sum_{i=1}^{m} C_i\,(||x_i - \theta||_i)^*$$

The basic difference between this problem and problem (T1) is that a norm is used in (T5) instead of a metric. If C_i is strictly increasing, this makes the problem strictly quasiconvex and thus guarantees that any local optimal solution is globally optimal. Although the existence of an optimal solution can be proved, the optimal solution surprisingly may not be in the convex hull of the points $\{x_1, \ldots, x_m\}$ (e.g. see [31]). (To see why this is surprising see the Appendix for a discussion of the relationship of convex hulls to compromises.). Finally, if $C_i(\cdot)$ is convex then problem (T5) becomes a convex programming problem to which duality theory can be applied. This will be discussed in problem (T8).

Besides having the above structure, problem (T5) has an interesting result in an important special case. In particular, we consider the case where n = 1 (e.g. R^n becomes the real line R^1) such that we are in the one dimensional issue space considered by Hotelling [14], Downs [8], and Tullock [30].

---

*Unlike the use of the superscript $||\cdot||^{(p)}$ to denote a particular ℓ_p norm, the use of the subscript $||\cdot||_i$ merely indicates that the type of norm is individual for each citizen.

Theorem 3.2 If $C_i(\cdot)$ are concave functions then the optimal location θ^* will be at some point x_i for a problem in R^1.

The proof follows from the observation that all norms are "equivalent" in R^1 and a property of concave functions (see [32]).

Even if the functions $C_i(\cdot)$ are not concave, we still have the following result.

Theorem 3.3 In R^1 problem (T5) will have an optimal location $\theta^* \in$ c.h. $\{x_1, \ldots, x_m\}$.*

Again the proof follows from the "equivalence" of norms in R^1.

By now the reader may suspect that we are examining a sequence of problems with different levels of abstraction. If so, he is right. The interrelationships of these various problems are given in Figure 3.1. The top problem represents the most abstract one with special cases being represented by arrows. Thus, any result for a problem will also be true for any of its special cases. For example, what is true of (T5) is also true of (T6), (T7), (T8), and (T9). On the other hand, we will sometimes

PLACE FIGURE 3.1 ABOUT HERE

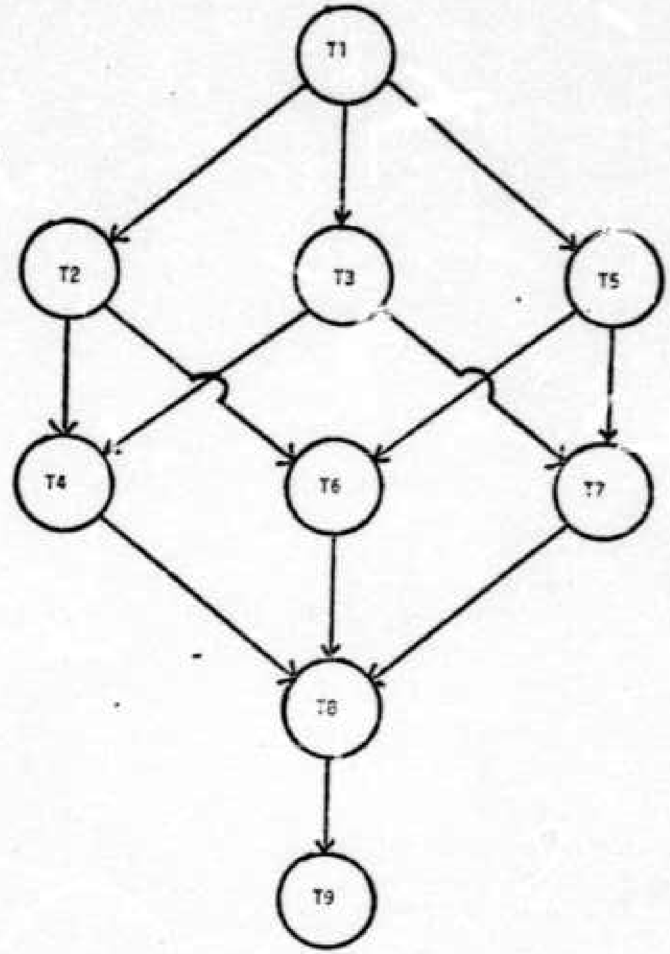
be able to say more about the special cases since they have more structure. This will become evident as we proceed.

We now consider a special case of problems (T2) and (T5).

$$\text{(T6)} \quad \min_{\theta} \sum_{i=1}^{m} w_i \, ||x_i - \theta||_i$$

*The notation c.h. $\{x_1, \ldots, x_m\}$ means the convex hull of the set $\{x_1, \ldots, x_m\}$. See the Appendix for further discussion.

12



Interrelationship of the Formulations

Figure 3.1

In particular, (T6) is a special case of (T2) in that each metric $\rho_i(\cdot\,,\cdot)$ is now assumed to be a norm $||\cdot||_i$. On the other hand, (T6) is a special case of (T5) if one lets $C_i(z) = w_i z$. The interpretation of such assumptions have been previously given when discussing problems (T2) and (T5).

Before giving a result for this problem, we first recall a basic property of norms.

<u>Lemma</u> (see page 132 of [25])).

Let $||\cdot||_1$ and $||\cdot||_2$ be two given norms. Then there exists $L_2 > 0$ and $U_2 > 0$ such that for every $\xi \in R^n$ we have

$$L_2\,||\xi||_2 \le ||\xi||_1 \le U_2||\xi||_2$$

For a given norm $||\cdot||_k$ we define

$$\overline{U}_i \equiv \text{Min}\,\{U_i : ||\xi||_i \le U_i\,||\xi||_k\} \text{ and}$$

$$\overline{L}_i \equiv \text{Max}\,\{L_i : ||\xi||_i \ge L_i\,||\xi||_k\}\,.$$ If, of course, all norms are of the same type, then $\overline{L}_i = \overline{U}_i = 1$. Although $\overline{U}_i \ge \overline{L}_i$ in general, it is possible that $\overline{U}_i \le 1$.

We now use this lemma to state a generalization of the Majority Theorem in the multi-norm problem (T6).

<u>Theorem 3.4</u> If $w_k \ge \sum_{\substack{i=1\\ i\ne k}}^{m} \overline{U}_i\, w_i$, then $\theta^* = x_k$.

<u>Proof</u> (see [35]).

Thus, if the norm being used by each citizen is known, the dictator can readily calculate $\overline{U}_1, \ldots, \overline{U}_m$ with respect to a given norm $||\cdot||_k$. Then, given weights w_i, a dictator should locate at x_k if $w_k \ge \sum_{\substack{i=1\\ i\ne k}}^{m} w_i\,\overline{U}_i$. In particular, if the "power" (e.g. loss) of the k^{th} citizen is greater than or

13

equal to the weighted sum of the "power" (i.e. loss) of all other citizens (with the weights $\overline{U}_i$ as defined), then the dictator should locate at x_k. Since $\overline{U}_i = 1$ if all norms are of the same type, the above theorem is a generalization of the Majority Theorem.

Now we look at a special case of problems (T3) and (T5):

$$\text{(T7)}\quad \text{Min} \sum_{i=1}^{m} C_i\,(||\,x_i - \theta||).$$

In particular, (T7) is a special case of (T3) in that the metric $\rho(\cdot,\cdot)$ is now replaced by a norm $||\cdot||$. It is a special case of (T5) in that (T7) assumes that all citizens have the same type of norm.

As given in (1.2), Davis, Hinich, and Ordeshook [5] not only assume that all citizens have the same type of norm but further assume that this norm is $[(x_i - \theta)'\,A\,(x_i - \theta)]^{1/2}$. With respect to generalizing their assumption they state (see page 434 of [5]) that "such a step results in a model whose complexity appears to prohibit the realization of meaningful analytical results. Accordingly, we use a simpler approach." This model (T7), and in fact this whole paper, is in direct contrast to their statement.

Along these lines we now point out how problem (1.2) is a special case of problem (T7). To see this, make the following identifications

$$||x_i - \theta|| = \left[(x_i - \theta)'\,A\,(x_i - \theta)\right]^{1/2}$$

$$C_i(z) = w(x_i)\,\phi\,(z^2)$$

so that $C_i\,(||x_i - \theta||) = w(x_i)\,\phi\,((x_i - \theta)'\,A\,(x_i - \theta))$.

Later we will further show how problem (1.4), which is equivalent to (1.2), is a special case of problem (T7.2), which in turn is a special case of problem (T7).

This problem has been considered in [31] and among the results is the

following.

<u>Theorem 3.5</u> If $n \leq 2$ then there exists an optimal solution $\theta^* \in$ c.h. $\{x_1, \ldots, x_m\}$.

In other words, if there are at most two issues then an optimal dictator location is in the convex hull of the citizen's positions (e.g. if there are three citizens, then there is an optimal dictator location in the triangle formed by the three points). This seems reasonable in that the dictator desires to reach a compromise among the citizens (see the Appendix for a further discussion on this point). <u>What is surprising is that when $n \geq 3$ (e.g. at least 3 issues) this convex hull property is not true (see [35]).</u> More specifically, the optimal dictator location may not be in the convex hull of $\{x_1, \ldots, x_m\}$ when we have more than two issues. Moreover, recall that this convex hull property is not true for problem (T5) even when $n = 2$.

Also, recall that for $n = 1$, we discussed the convex hull property in Theorem 3.3. We now consider a more general case where the positions $\{x_1, \ldots, x_m\}$ are collinear (e.g. lie on a straight line L) in R^n. More specifically, we allow the issue space to be multi-dimensional but we consider the case where the points $\{x_1, \ldots, x_m\}$ are collinear in that space. Note that if $n = 1$ (e.g. we are in R^1) then the points $\{x_1, \ldots, x_m\}$ must be collinear. Thus, our present case is a generalization of the R^1 case in that we consider the preferences of citizens on positions that do not lie on the line L. With respect to this more general situation, we get the following result that reduces the problem to one of location on the line L.

<u>Theorem 3.6</u> In problem (T7) consider the case where the points $\{x_1, \ldots, x_m\}$ lie on some line L. Then for every $\theta \notin L$ there exists a $\bar{\theta} \in L$ such that

$$\sum_{i=1}^{m} C_i\left(\|x_i - \bar{\theta}\|\right) \leq \sum_{i=1}^{m} C_i\left(\|x_i - \theta\|\right) .$$

14

<u>Pf.</u> (see [35]).

This theorem says that the dictator need only consider points on the line L as possible optimal locations. Thus the problem "essentially reduces" (see [31] for details) to one in R^1 and Theorems 3.2 and 3.3 apply. We can, therefore, state the following result.

<u>Corollary</u> If $\{x_1, \ldots, x_m\}$ are collinear in R^n then an optimal location for problem (T7) in the convex hull of $\{x_1, \ldots, x_m\}$ will exist. Furthermore, if $C_i(\cdot)$ are concave functions, then some x_i will be optimal.

The surprising thing about the above result is that it depends on all citizens having the same type of norm. If, for example, the points $\{x_1, \ldots, x_m\}$ in problem (T5) are collinear, the result is not necessarily true.

<u>Example 3.1</u>

Consider the case of two citizens with locations $x_1 = \begin{pmatrix}2\\0\end{pmatrix}$ and $x_2 = \begin{pmatrix}0\\1\end{pmatrix}$ in R^2. Suppose that citizen x_1 uses the ℓ_∞ norm while citizen x_2 uses the ℓ_1 norm. If the loss of each citizen is proportional to distance with weights $w_1 = 3/2$ and $w_2 = 1$ for citizens x_1 and x_2 respectively, then the dictator's problem becomes:

$$\min_{\theta} \; 3/2\|x_1 - \theta\|^{(\infty)} + \|x_2 - \theta\|^{(1)} .$$

It is easy to verify that every point on the line L through x_1 and x_2 has a strictly higher total loss than the point $\begin{pmatrix}1\\1\end{pmatrix}$. Thus, the optimal location in this case is not on the line L and the problem does not "reduce" to the R^1 case.

The above example and the previous theorems lead to the following conclusion: <u>If the points $\{x_1, \ldots, x_m\}$ are collinear in R^n and if all citizens</u>

have the same norm in problem (T5) (e.g. as in problem (T7)), then the dictator's location problem becomes "equivalent" to a simple location problem in R^1. In general, however, such a reduction in problem (T5) is not possible. Therefore, a careful distinction must be made in problem (T5) as to whether the points $\{x_1, \ldots, x_m\}$ are collinear in R^n or whether the issue space is R^1.

We will now look at the three important special cases of (T7) when the norms are $\ell_1, \ell_2, \ell_\infty$ respectively. We denote these special cases as problems (T7.1), (T7.2), and (T7.∞).

$$\text{(T7.1)} \quad \min_{\Theta} \sum_{i=1}^{m} C_i \left(\| x_i - \Theta \|^{(1)} \right)$$

This problem is simply one of minimizing total utility loss when the distance measure for all citizens is the ℓ_1 norm. If C_i is concave we get the interesting result that the optimal solution Θ^* is at one of a finite number of "intersection points." We will not rigorously define an "intersection point" here (see [21]), but instead we illustrate the concept in Figure 3.2. In R^2 the intersection points are given by the intersection of lines parallel to the axes through each point x_i for $i = 1, \ldots, m$. In R^3 we use planes instead of lines and in general we use hyperplanes*.

<u>Theorem 3.7</u> Let $\{\gamma_1, \ldots, \gamma_k\}$ be the set of intersection points of the set $\{x_1, \ldots, x_m\}$ where $x_i \in R^n$. Then $k \leq m^n$. If C_i is a concave function for $i = 1, \ldots, m$, then $\Theta^* \in \{\gamma_1, \ldots, \gamma_k\}$.

Thus, if each citizen's loss function $C_i(\cdot)$ is concave then some intersection point will be an optimal location for the dictator.

PLACE FIGURE 3.2 HERE

*See [23] for a definition of a hyperplane.

15

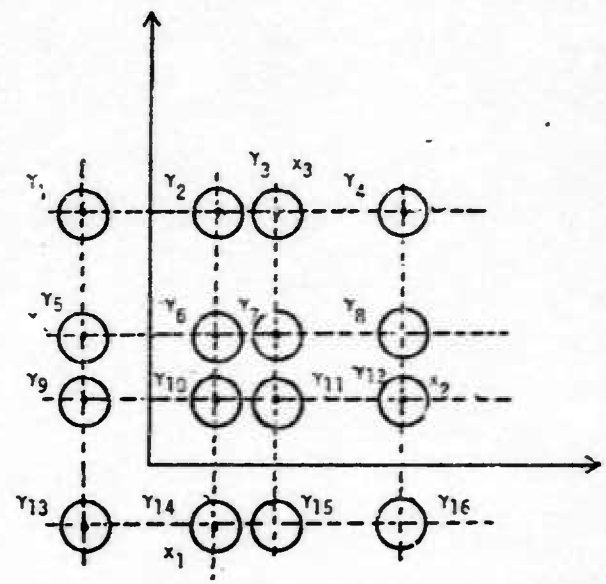


Figure 3.2

We now consider the problem (T7) under the familiar Euclidean norm.

$$\text{(T7.2)} \quad \operatorname*{Min}_{\theta} \sum_{i=1}^{m} C_i \left(\| x_i - \theta \|^{(2)} \right)$$

It is interesting that the optimal solution(s) for this problem has a property which (as we have pointed out) is not true for (T7) in general.

Theorem 3.8 For arbitrary n there exists an optimal solution $\theta^* \in$ c.h. $\{x_1, \ldots, x_m\}$. See [31] for a proof of this result.

Note that by letting

$$C_i(z) \equiv w(x_i) z^2$$

problem (T7.2) reduces to the problem (1.4).

Since (1.4) and (1.2) are equivalent (see section 1 for a discussion), Theorems 3.8 also applies to problem (1.2).

Recall that Davis, Hinich, and Ordeshook [5] obtained a much more specific result for their problem that what we have so far obtained. In particular, by by making various symmetry, unimodality, and marginally increasing assumptions (e.g. see section 1), they have shown that the mean is the optimal dictator location. We have made no such assumptions and have, of course, not arrived at such a specific result. Later, however, as we make more explicit assumptions, we will show that median also can be a natural optimal location for a dictator. Such a result is parallel to the one-dimensional analyses of democracies (see [8], [14], [30]) as well as to the generalizations of this one-dimensional analysis that we give in [34].

Under the ℓ_∞ norm problem (T7) becomes

16

$$\text{(T7.}\infty\text{)} \quad \operatorname*{Min}_{\theta} \sum_{i=1}^{m} C_i \left(\| x_i - \theta \|^{(\infty)} \right).$$

Recall that in a two dimensional issue space R^2 our theorem in section 2 showed that the $\|\cdot\|^{(\infty)}$ and $\|\cdot\|^{(1)}$ norms are equivalent under a 45 degree rotation. Hence, our previous theorem for optimal location to (T7.1) at intersection points also applies to (T7.∞) after such a rotation if we are in R^2. For R^n with $n \geq 3$ this problem has not been studied (probably because of its lack of applications to plant location theory). A special case of (T7.∞) when $C_i(z) = w_i z$ can be solved and will be considered in our discussion of problem (T8).

We now consider an important special case of problems (T4), (T6), and (T7).

$$\text{(T8)} \quad \operatorname*{Min}_{\theta} \sum_{i=1}^{m} w_i \| x_i - \theta \|$$

This problem is a special case of (T4) in that a norm is used instead of a metric as a distance measure. From (T6) we get (T8) by assuming that all norms are of the same type. Finally, from (T7) we get (T8) by assuming that $C_i(z) = w_i z$.

Before going into the special cases of (T8) that arise from considering specific norms, we first consider the case where $\{x_1, \ldots, x_m\}$ are collinear along same line L. Besides having the results from Theorem 3.6 and its Corollary, we further have the following result.

Theorem 3.9 Suppose that in problem (T8) the points $\{x_1, \ldots, x_m\}$ are collinear along some line L. For this line L specify some direction of increase so that we can relate any two points x_i, x_k in the line L by the relationships $\geq$, $\leq$, $>$, $<$, and $=$.

Then x^* on L is an optimal location iff

$$\sum_{i:\ \{x_i < x^*\}} w_i \le \sum_{i:\ \{x_i \ge x^*\}} w_i$$

and

$$\sum_{i:\ \{x_i \le x^*\}} w_i \ge \sum_{i:\ \{x_i > x^*\}} w_i .$$

Furthermore (by Theorem 3.6), these conditions will be satisfied for some $x^* = x_k$.

In the above theorem, the optimal location of the dictator corresponds to a point x^* where the sum of the slopes of the citizens' loss functions to the "left" of the dictator is counterbalanced by the sum of the slopes of the citizens' loss functions to his "right." Roughly speaking, x^* is a point where the "power" of citizens to the "right" are counterbalanced by citizens to the "left." Some x_k will always have this property. Furthermore, when all w_i are equal (or equivalently, when they all equal one), x^* is just the median of the points $\{x_1, \ldots, x_m\}$ along the line L. Again we consider the three special cases of the ℓ_p norm: the ℓ_1 norm $||\cdot||^{(1)}$, the ℓ_2 norm $||\cdot||^{(2)}$, and the ℓ_∞ norm $||\cdot||^{(\infty)}$.

First, under the ℓ_1 norm, we have the following problem.

$$\text{(T8.1)} \quad \underset{\theta}{\text{Min}} \sum_{i=1}^{m} w_i \, || x_i - \theta ||^{(1)}$$

This nonlinear problem has the interesting property of possible reformulation as a linear program [36]. As a linear program the optimal solution is easily

17

determined via the Simplex method. Although such a reformulation can be made in general for R^n, for simplicity we illustrate the approach for problems in R^2.

Example 3.2

In R^2 problem (T8.1) becomes

$$\underset{\theta^1, \theta^2}{\text{Min}} \sum_{i=1}^{m} w_i \left(|x_i^1 - \theta^1| + |x_i^2 - \theta^2| \right)$$

By introducing other variables and exploiting properties of absolute values, we get the problem.

$$\underset{\substack{\theta^1, \theta^2, y_i^1, y_i^2 \\ i = 1, \ldots, m}}{\text{Min}} \sum_{i=1}^{m} w_i y_i^1 + w_i y_i^2$$

s.t.

$$y_i^1 \ge x_i^1 - \theta^1$$
$$y_i^1 \ge \theta^1 - x_i^1$$
$$y_i^2 \ge x_i^2 - \theta^2$$
$$y_i^2 \ge \theta^2 - x_i^2 \qquad \text{for } i = 1, \ldots, m.$$

The above problem is a linear program whose optimal solution can be readily determined via the Simplex method.

Although the above example illustrates a method of finding the optimal solution, the process provides little insight into the nature of the optimal solution. Recall, however, that from our study of problem (T7.1) we pointed out that an optimal location will exist at some intersection point. We can now strengthen this result through the following theorem [37].

First a relatively simple result is given.

Theorem 3.11 If the points x_i are not collinear then there exists a unique optimal solution. The proof [19] follows from the fact that $\sum_{i=1}^{m} w_i \, ||x_i - \theta||^{(2)}$ is strictly convex.

Also, the following result is quite remarkable.

Theorem 3.12 Let $\phi \equiv \underset{\theta}{\mathrm{Min}} \sum_{i=1}^{m} w_i \, ||\theta - x_i||^{(2)}$ be called the orimal problem. Now consider the dual program to the above problem*.

$$\chi = \underset{y_1, \ldots, y_m}{\mathrm{Max}} \sum_{i=1}^{m} \langle x_i, y_i \rangle$$

$$\text{s.t.} \quad ||y_i||^{(2)} \leq w_i \text{ for } i=1, \ldots, m$$

$$\sum_{i=1}^{m} y_i = 0$$

Then $\phi = \chi$ and the optimal solutions to the above problems are related as follows:

$$y_i^* = \begin{cases} \dfrac{w_i (x_i - \theta^*)}{||\theta^* - x_i||^{(2)}} & \text{if } \theta^* \neq x_i \\ -\sum\limits_{\substack{k=1 \\ k \neq i}}^{m} y_k^* & \text{if } \theta^* = x_i, \text{ for } i=1, 2, \ldots, m^{**}. \end{cases}$$

*The inner product $\langle x_i, y_i \rangle$ has its usual definition as $\sum_{j=1}^{n} x_i^j y_i^j$.

**Since θ^* will equal at most one x_i, no confusion concerning these cases exists.

18

See [19] for further discussion and a proof of the above duality theorem. For a generalization of the theorem to problem (T8) see [37]. Also, for a generalization to problem (T5) see [33]. For our purposes of understanding and discussing the concept of duality, we limit ourselves to problem (T8.2) and the above theorem.

In physics the primal problem can be interpreted as finding the point of minimum potential energy. One can think of this by considering the problem in R^2 and by imagining a smooth (frictionless) table top with holes at the points x_i. Then we get m strings, put the i^{th} string through the hole x_i, and attach a weight w_i to the end of the string hanging under the table. Finally, we tie all m strings together into one knot that is free to move on the top of the table. Since there is no friction, the knot will come to rest at the optimal location θ^*.

In physics there is a duality between potential energy and force. In particular, we can view the problem of minimizing potential energy as one of finding those set of forces $\{y_1^*, \ldots, y_m^*\}$ along the strings 1, ..., m respectively that sum to zero (e.g. are in equilibrium). The dual problem solves for this set of optimal forces.

The interpretation of these forces in political terms is especially interesting. In particular, a dictator located at θ^* will "feel" a force on him y_i^* for i=1, ..., m. Unless $\theta^* = x_i$, the magnitude of this force will be w_i.

Since $\theta = \chi$ (i.e. no duality gap) and since the relationship between the optimal primal and dual variables are necessary and sufficient, one can derive the following theorem for optimal locations (see [33]).

Theorem 3.12 A solution θ (not equal to any x_i) to the primal is optimal iff

$$\sum_{i=1}^{m} w_i \frac{(x_i - \theta)}{||\theta - x_i||^{(2)}} = 0 .$$

Equivalently, letting e_j be the j^{th} unit vecter in R^n, this condition can be written as

$$\sum_{i=1}^{m} w_i \cos \langle e_j , x_i - \theta = 0 \quad \text{for } j=1, \ldots , n^*.$$

Note that the cosine terms in the above condition takes account of the relative positions of the points x_i in R^n.

Alternately, for a solution $\theta = x_i$, we have the following result:

Theorem 3.14 A solution $\theta = x_i$ is optimal iff

$$\left|\left| \sum_{\substack{k=1 \\ k \neq i}}^{m} w_k \frac{(x_k - \theta)}{||\theta - x_k||^{(2)}} \right|\right|^{(2)} \leq w_i .$$

It is perhaps easiest to interpret the above theorem in terms of forces. In particular, if the dictator locates at some point $\theta = x_i$, then the force that citizen x_k (for $k \neq i$) is $\frac{w_k (x_k - \theta)}{||\theta - x_k||^{(2)}}$.

The vector sum of all such forces is

$$\sum_{\substack{k=1 \\ k \neq i}}^{m} w_k \frac{(x_k - \theta)}{||\theta - x_k||}^{(2)}$$

and this sum has the magnitude given by the left expression in Theorem 3.14. On the other hand, the magnitude of the force extended by citizen x_i is w_i. Thus, the theorem says that x_i is optimal iff the force w_i is greater than or equal to the magnitude of the sum of forces of all other citizens.

Again, the relative positions of the points x_i play an important role in the above result. Using the triangle inequality for norms we can easily derive the following sufficient condition which is true regardless of the relative position of the points x_i.

Corollary

If $\sum_{\substack{k=1 \\ k \neq i}}^{m} w_k \leq w_i$ then $\theta^* = x_i$ . .

This, of course, is identical to the Majority Theorem 3.1, which we gave for a much more abstract problem (e.g. see problem (T4)). Comparing Theorem 3.14 to Theorem 3.1 one can readily see how the additional structure in problem (T8.2) over problem (T4) leads to a more powerful result.

We now terminate our discussion of problem (T8) by considering the case of the ℓ_∞ norm.

$$(T8.\infty) \quad \underset{\theta}{\text{Min}} \sum_{i=1}^{m} w_i \, || x_i - \theta ||^{(\infty)}$$

As discussed previously, the results for (T8.1) and (T8.∞) are "equivalent" in R^2 after one makes a change of variables to effect a 45 degree rotation. We will not dwell on the details of this "equivalence." Instead we will

*cos $\langle e_j , x_i - \theta$ is the cosine of the angle between the vectors e_j and $x_i - \theta$.

illustrate how this problem can be reformulated as a linear program. Although such a reformulation can be made in R^n, for simplicity we consider the problem in R^2.

Example 3.3

In R^2 problem (T8.∞) becomes

$$\underset{\theta^1,\,\theta^2}{\text{Min}} \sum_{i=1}^{m} w_i \;\text{Max}\;\{|\,x_i^1 - \theta^1|\;,\;|\,x_i^2 - \theta^2\,|\,\} \quad .$$

Using various absolute value "tricks", we get the problem

$$\underset{\substack{\theta^1,\,\theta^2,\,y^i \\ i=1,\,\dots,\,m}}{\text{Min}} \sum_{i=1}^{m} w_i\, y_i$$

s.t.

$$y_i \geq x_i^1 - \theta^1$$
$$y_i \geq \theta^1 - x_i^1$$
$$y_i \geq x_i^2 - \theta^2$$
$$y_i \geq \theta^2 - x_i^2 \quad .$$

for $i = 1, \dots, m$.

This is a linear program which can be solved via the Simplex method.

Finally, we come to the last problem in our heirarchical classification.

$$\text{(T9)} \qquad \underset{\theta}{\text{Min}} \sum_{i=1}^{m} \|\, x_i - \theta \|$$

Note that (T9) is a special case of (T8) that results when $w_i = 1$ for $i=1, \dots, m$ (e.g. or, in general, when $w_1 = w_2 = \dots = w_m$).

Again, the collinear case, gives us an important special result.

20

Theorem 3.15 If $\{x_1, \dots, x_m\}$ are collinear on some line L then x^* is the median of the points $\{x_1, \dots, x_m\}$ over the line L.

The proof follows from our discussion following Theorem 3.9

Another result which also illustrates the basic importance of the median is the following theorem for the non-collinear case where the norm is of the ℓ_1 type.

Theorem 3.16 In problem (T9.1) the optimal solution θ^* is at the multi-dimensional median. More specifically, the j^{th} coordinate θ^{j*} of θ^* is the median of the points $\{x_1^j, x_2^j, \dots, x_m^j\}$. (The proof of this this theorem follows from Theorem 3.10).

Thus, the dictator adopts a median position on each of the issues. Note that the position on the issues are independent of one another. His optimal location at the (multi-dimensional) median is in direct contrast to his location at the mean in [5]. There is no question of who is more correct. One can only ask what set of assumptions apply to the particular situation being considered.

As per problem (T9.∞) we merely point out that after rotating the position 45 degrees, the multi-dimensional median result of the ℓ_1 norm case applies.

Finally, for problem (T9.2) we get an interesting graphical solution procedure that yields the optimal solution to the problem in R^2. Instead of reviewing this graphical technique here, we simply refer the reader to [19]. Note that this technique does not have any direct extension to the case R^3 or to the case of weighted distance (e.g. problem (T8.2)). Thus, various

papers have been written to handle these more general cases (for example, see [18], [20], and [21]).

For a summary of the main results in this section see Figure 3.3 which associates theorems with the problem structure illustrated in Figure 3.1.

PLACE FIGURE 3.3 HERE

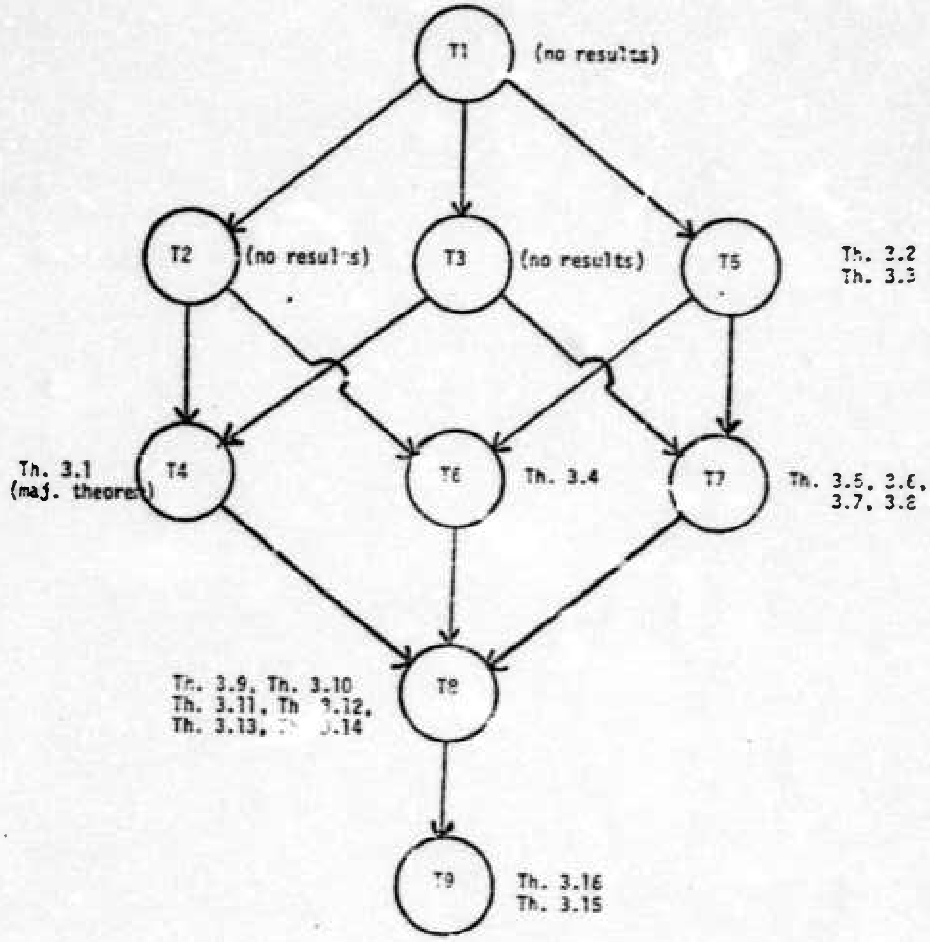


Figure 3.3

4. Minimizing Maximum Utility Loss

Recently in location theory the problem of locating emergency facilities (e.g. hospitals, fire stations, etc.) has been considered. Essentially, the criterion used is to minimize the maximum of all possible distances (e.g. time) to some disaster. This problem is in some respects similar to the one considered in [5] -- only here concern is with extremists. That is, the dictator wants to find some position, θ^*, which minimizes the maximum loss felt by any citizen as a result of his [the dictator's] position. As will be shown, this objective on the part of the dictator leads to different optional locations than does the "minimize total utility loss" objective considered in the previous section. Thus, the problem has a typical formulation as

$$(M9) \qquad \underset{\theta}{\text{Min}} \quad \underset{i \in \{1,\ldots,m\}}{\text{Max}} \quad ||x_i - \theta|| \quad .$$

Note that the above problem (M9) is similiar to problem (T9) with the exception that the summation operator $\sum_{i=1}^{m}$ is now replaced by the maximizing operator $\underset{i \in \{1,\ldots,m\}}{\text{Max}}$. Making this replacement in each of the problems (T1) through (T9) we get a new set of problems (M1) through (M9). Furthermore, the problems (M1) through (M9) have exactly the same hierarchical structure as problems (T1) through (T9) as given in Figure 3.1. We will not explicitly discuss problems (M1) through (M7), but instead we will consider the problems (M8) and (M9). The reason for this is that not much is known about problems (M1) through (M7). (The theorems of convex programming can, however, be applied to problems (M5) through (M9)). On the other hand, problem (M9) falls into a class of problems generally known as Spherical

22

Covering Problems. As with the Weber problem, this type of problem has a long and interesting history (see [9] for a discussion).

Its relation to the dictator problem is especially interesting and, as we have seen above, leads to a whole new class of problems. In particular, the problem is one of a dictator who wants to pick a position θ^* that minimizes the maximum utility loss (e.g. "unhappiness") of all the citizens. Such a criterion clearly caters to the citizens who are at extreme positions on issues.

Assuming the Euclidean norm, the dictator's problem (M9) becomes

$$(M9.2) \qquad \underset{\theta}{\text{Min}} \quad \underset{i \in \{1, \ldots, m\}}{\text{Max}} \quad || x_i - \theta||^{(2)} \quad .$$

Geometrically this can be interpreted as finding the center of the smallest sphere (e.g. circle or ball) that encloses the points $\{x_1, x_2, \ldots, x_m\}$. (For problem (M8.2) we must consider using an ellipse.). In R^2 the solution is immediate from elementary plane geometry. In general, the problem is not so simple. For further discussion on this point we refer the reader to [9].

Assuming the ℓ_1 norm, we get the problem

$$(M9.1) \qquad \underset{\theta}{\text{Min}} \quad \underset{i \in \{1, \ldots, m\}}{\text{Max}} \quad || x_i - \theta||^{(1)}$$

Similar to (M9.2) this problem can be interpreted as finding the center of the smallest diamond enclosing the points $\{x_1, \ldots, x_m\}$. For a further discussion of the geometry of this problem see [11]. Furthermore, this problem can be reformulated as a linear program [6]. We will now consider such a reformulation for the more general problem (M8.1). In particular, we illustrate the reformulation for a problem in R^2.

Example 4.1 In R^2 problem (M9.1) becomes

$$\min_{\theta^1,\theta^2} \max_{i\in\{1,\dots,m\}} w_i \left(|x_i^1 - \theta^1| + |x_i^2 - \theta^2| \right) .$$

As before we can introduce new variables to get the following linear programming problem*.

$$\min_{\theta^1,\theta^2,\xi} \xi$$

s.t.

$$\frac{\xi}{w_i} + \theta^1 + \theta^2 \geq + x_i^1 + x_i^2$$

$$\frac{\xi}{w_i} + \theta^1 - \theta^2 \geq x_i^1 - x_i^2$$

$$\frac{\xi}{w_i} - \theta^1 + \theta^2 \geq - x_i^1 + x_i^2$$

$$\frac{\xi}{w_i} - \theta^1 - \theta^2 \geq - x_i^1 - x_i^2$$

for $i=1, \dots, m$.

It is interesting to note that for problem (M9.1) we can obtain a further simplification to the following problem with three variables and four constants.

$$\min_{\theta^1,\theta^2,\xi}$$

s.t.

$$\xi + \theta^1 + \theta^2 \geq b_1$$

---

*We can assume without loss of generality that $w_i > 0$.

23

$$\xi + \theta^1 - \theta^2 \geq b_2$$
$$\xi - \theta^1 + \theta^2 \geq b_3$$
$$\xi - \theta^1 - \theta^2 \geq b_4$$

Now we come to the problem

$$\text{(M9.}\infty\text{)} \quad \min_{\theta} \max_{i\in\{1,\dots,m\}} \| x_i - \theta \|^{(\infty)} .$$

This problem has a geometric interpretation as finding the center of the smallest cube (e.g. square in R^2) containing the points $\{x_1, \dots, x_m\}$. In the more general problem,

$$\text{(M8.}\infty\text{)} \quad \min_{\theta} \max_{i\in\{1,\dots,m\}} w_i \| x_i - \theta \|^{(\infty)}$$

we can imagine using rectangles instead of squares, etc.

We will now consider a reformulation of the above problem. For simplicity, we again illustrate the process in R^2.

Example 4.2. In R^2 problem (M8.∞) becomes

$$\min_{\xi,\theta^1,\theta^2} \max_{i\in\{1,\dots,m\}} w_i \max\left[|x_i^1 - \theta^1|, |x_i^2 - \theta^2|\right]$$

Using the usual substitutions, it is easy to verify that this is equivalent to the linear program

$$\min_{\xi,\theta^1,\theta^2} \xi$$

s.t.

$$\xi + w_i\theta^1 \geq w_i x_i^1$$
$$\xi - w_i\theta^1 \geq - w_i x_i^1$$
$$\xi + w_i\theta^2 \geq w_i x_i^2$$
$$\xi - w_i\theta^2 \geq - w_i x_i^2$$

for $i=1,\dots,m$.

For the problem (H9.=) (e.g. $w_i = 1$ for all i), we can obtain the simpler problem

$$\min_{\xi,\,\theta^1,\,\theta^2} \xi$$

$$\text{s.t.}\quad \begin{aligned} \xi + \theta^1 &\geq b_1 \\ \xi - \theta^1 &\geq b_2 \\ \xi + \theta^2 &\geq b_3 \\ \xi - \theta^2 &\geq b_4 \end{aligned}$$

24

5. Conclusions and Observations

This paper has oresented a methodology that can be applied to all levels of group decision making--from small committees to countries. In part it is a generalization of some recent work in spatial theory (e.g. [5], [24]). Our general point of view about this previous work is that although it blazes a new and exciting path, its models are not general enough to capture many important aspects of the "real world." To overcome this difficulty we have defined a heirarchical classification of problems with various levels of abstraction into which their problem becomes a very special case.

Such an approach is important since the higher the level of abstraction, the less one need assume about the structure of the problem. On the other hand, the more that one is prepared to assume about the structure of the problem, the stronger the assertions he is able to make about behavior in such a problem. One good example of this point comes from considering problems (T4) and Theorem 3.1 versus problem (T8.2) and Theorem 3.12.

One of the principle reasons for the generality in this paper is the approach used to characterize preference of the citizens. This point is essentially one of considering various ways that a citizen might perceive the "distance" between two positions in an issue space. Rather than limit ourself to the classic Euclidean distance measure, we introduced the mathematical concept of a metric and a norm from Functional Analysis. The generality of this approach is evident from the fact that there is a one-to-one correspondence between norms and all possible convex symmetric indifference

contours. Furthermore, we pointed out how two of these norms (e.g. ℓ_1 and ℓ_∞) may be just as important (if not more so) than the Euclidean distance norm. These two norms have some important properties which are reflected in the theorems and the examples. Although we cannot cover these important properties in this section, one which stands out is the new significance given to the "median" as an optimal position as opposed to the mean. This is a phenomena which also surfaces in our analysis of democracies [34] in which we extend the one-dimensional results of Hotelling [14], Downs [8], and Tullock[30].

Besides generalizing the concept of distance perception in an issue space, we also considered two different but perfectly reasonable objectives that a benevolent dictator might have. The first objective, which follows [5], is to assume that the dictator wishes to minimize total utility loss. This is considered in section 3. On the other hand, section 4 considers the possibility that the dictator might want to minimize maximum utility loss of the citizens. This is a new critereon which gives somewhat different results for his optimal location.

Finally, we wish to conclude with the observation that the results and methodology of this paper raise a number of additional theoretical and application questions. We leave it to the reader's imagination as to what forms such questions should take.

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26

APPENDIX

Convexity and Optimal Solutions

In this section we review some of the mathematical concepts used in this paper*. These concepts are well known in the optimization literature and we refer the reader to [23] for a further discussion of them.

Consider a set S of points in R^n.

Definition: The set S is said to be convex if for arbitrary $x_1 \in S$, $x_2 \in S$ and $0 \leq \lambda \leq 1$ we have $\lambda x_1 + (1 - \lambda) x_2 \in S$.

That is, in an issue space R^n, a set of positions S is convex if for any two positions in S their weighted average (with positive weights that sum to one) is also in S. Geometrically this means that a set S is convex if for any two points x_1 and x_2 in the set we have all points on the line segment connecting x_1 and x_2 also in the set.

We now give an equivalent condition for a set to be convex.

Theorem A set S is convex iff for any K points $\{x_1, \ldots, x_K\}$ from S we have that

$$\sum_{k=1}^{K} \alpha_k \; x_k \in S \qquad \text{where } \alpha_k \geq 0 \text{ and } \sum_{k=1}^{K} \alpha_k = 1.$$

Since when K=2 the above condition reduces to the usual definition of a convex set, it is at first surprising to see that the two conditions are equivalent.

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*The notation in this Appendix is somewhat different than the notation in other sections of this paper and the two should not be confused.

One of the reasons for giving the above theorem is to help the reader obtain a better understanding of the definition of a convex hull.

Definition: The convex hull of a set S, denoted as c.h.S, is the set $\{x : x = \sum_{k=1}^{K} \alpha_k x_k, \alpha_k \geq 0, \sum_{k=1}^{K} \alpha_k = 1, x_k \in S \text{ for } k = 1, \ldots, K\}$.

In other words, the convex hull of S is the set of all of those vectors x that can be expressed as $\sum_{k=1}^{K} \alpha_k x_k$ for some $\alpha_1, \ldots, \alpha_K \geq 0$, $x_1, \ldots, x_K \in S$ where $\sum_{k=1}^{K} \alpha_k = 1$. Equivalently, we give another characterization of a convex hull.

Theorem The convex hull of a set S, denoted as c.h.S, is the intersection of all the convex sets containing S.

In particular, it can be thought of as the smallest convex set containing S. Given a set of citizen's positions $\{x_1, \ldots, x_m\}$, the convex hull of $\{x_1, \ldots, x_m\}$ might be regarded as the set of "direct compromises" among these positions. This is illustrated in the following example.

Example

In the case when m = 3 suppose that the positions of x_1, x_2, and x_3 are given by the vectors $\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$ respectively. Figure A.1 depicts this case. The convex hull of these three points is the triangle (including its interior) formed by them. From the theory of convexity we assert that any point y in this triangle can be expressed as follows:

$$y = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

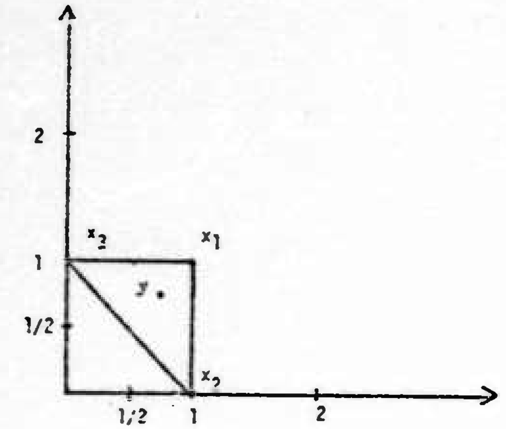


Figure A.1

For example, $\begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1/4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1/4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

PLACE FIGURE A.1 ABOUT HERE

In this case, one might interpret the point $\begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix}$ as the position resulting when citizen x_1 gets 1/2 of his own way while citizens x_2 and x_3 each get 1/4 of their own way. Thus the point 3/4 can be viewed as a "direct compromise" among citizens x_1, x_2, and x_3 with relative weights 1/2, 1/4, and 1/4 respectively.

Definition: An extreme point x of a convex set S is a point having the following property: for any $x_1 \in S$, $x_2 \in S$, and $0 < \lambda < 1$ where $x = \lambda x_1 + (1 - \lambda) x_2$ we have $x_1 = x_2 = x$.

For example, the extreme points of a rectangle are its four corners and the extreme points of a circle are the points on its circumference. In Figure A.1 the extreme points of the triangle are the points x_1, x_2, and x_2.

Definition: Consider a function $f(x)$ defined on a subset S of R^n into R^1. A mathematical program* is an optimation problem of the form

$$\text{(MP)} \qquad \underset{x \in S}{\text{Min}} \; f(x) \quad .$$

Definition: A point x^* is said to be the global optimal solution (e.g. global minimum) of (MP) if $f(x^*) \leq f(x)$ for all x in S.

*Although a more mathematically correct definition is to use infimum (Inf) instead of Min, we will not worry about the distinction here.

Because of the possibility of various nonlinearities in $f(x)$ and/or the possibility that S may not be convex, we further define another type of optimal solution.

Definition: A point x^* is said to be a local optimal solution (e.g. local minimum) if there exists some $\delta > 0$ such that $f(x^*) \leq f(x)$ for all x in the set $S \cap \{x : \|x-x^*\| \leq \delta\}$.

In certain important classes of mathematical programs, various properties about the optimal solution can be asserted. To characterize such classes of programs we now make the following definition.

Definition: Consider a function $f(x)$ defined on a convex set S in R^n. Such a function is said to be concave if for every $x_1 \in S$, $x_2 \in S$, and $0 \leq \lambda \leq 1$ we have $f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$. Alternately, the function is said to be convex if for every $x_1 \in S$, $x_2 \in S$, and $0 \leq \lambda \leq 1$ we have $f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$. If for every $x_1 \neq x_2$ and $0 < \lambda < 1$ we further have $f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2)$, then $f(x)$ is said to be strictly convex. (Similarly, we can define strictly concave.)

Roughly speaking, convex functions can be thought of as "U" shaped and concave functions as upside-down "U" shaped.

When S is convex and $f(x)$ is concave over S in (MP), the resulting problem is called a concave program. The following is a basic result for concave programs.

> Theorem If (MP) is a concave program for which an optimal solution exists*, then an optimal solution occurs at some extreme point of S.

*We avoid some of mathematical properties of the set S (e.g. compact).

From the above theorem we conclude that in concave programs it is sufficient to look for the global optimal solution over the set of extreme points. If, for example, $S = \{x : Ax \leq b\}$ where A is an $m \times n$ matrix and $b \in R^m$, then only a finite number of such extreme points exist.

If, on the other hand, $f(x)$ is convex over a convex set S in (MP), then the resulting problem is called a convex program. One of the main results for convex programs is the following.

> Theorem If (MP) is a convex program, then any local optimal solution is also a global optimal solution. Furthermore, if $f(x)$ is strictly convex, then the global optimal solution is unique.

Many of the problems in location theory turn out to be convex programs. Thus, in particular, the above theorem applies. There are, however, a number of other theorems which could be brought to bear on such problems. Among these theorems, the ones concerning Lagrangian multipliers and duality are especially interesting. Although we will not discuss these theorems here, we will include some discussion of their application in this paper.

In some important special cases (e.g. as we will see) location problems can be reformulated as linear programs (e.g. when $f(x) = \sum_{j=1}^{n} c^j x^j$ and $S = \{x : Ax \leq b\}$). Linear programs have the unique property of being both convex and concave programs. Thus, it is only necessary to search for the optimal solution over the finite set of extreme points until you find one which is locally (and therefore globally) optimal. One well known method for doing this is the Simplex algorithm. Also important in linear programming is the ability to perform post-optimal sensitivity and parametric analyses. For a further discussion of this, we refer the reader to any

standard linear programming book (e.g. see [12]).

For some problems it is important to characterize other classes of functions $f(x)$ in order to exploit their properties. We will only consider one such class.

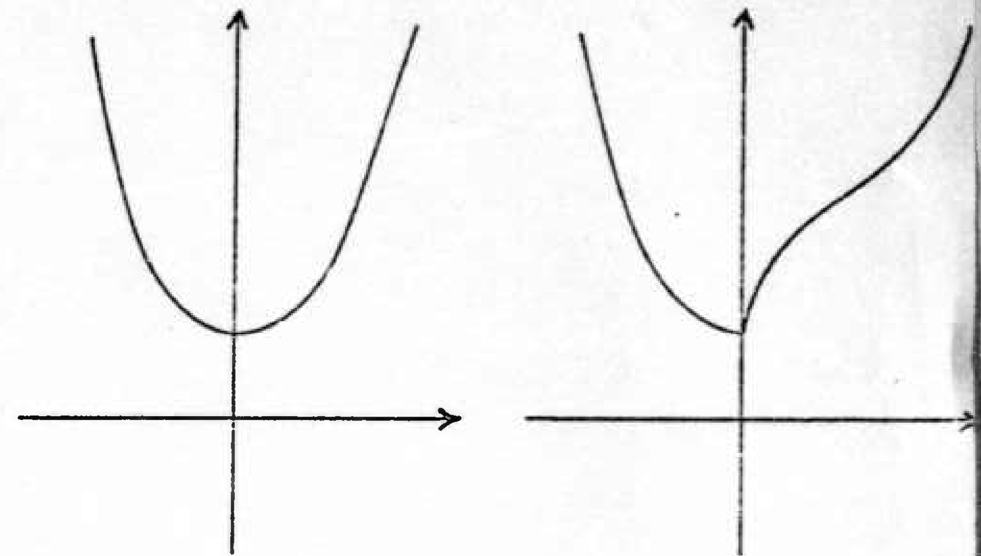
<u>Definition</u>: A function $f(x)$ is said to be <u>strictly quasiconvex</u> over the convex set S when for every $x_1, x_2 \in S$ and $0 < \lambda < 1$ where $f(x_2) < f(x_1)$ we have $f((1-\lambda)\, x_1 + \lambda\, x_2) < f(x_1)$.

Figure A.2 illustrates a strictly quasiconvex function and contrasts it to a (strictly) convex function. Note that the strictly quasiconvex function does not have to be convex. The importance of such functions

PLACE FIGURE A.2 ABOUT HERE

follows from the following theorem (which is a generalization of the previous theorem.

<u>Theorem</u> If $f(x)$ is a strictly quasiconvex function over the convex set S in (MP), then any local optimal solution is a global optimal solution.



<u>(Strictly) Convex Function</u> <u>Strictly Quasiconvex Function</u>

Figure A.2

30